Introduction

Welcome to Friday practices! I have prepared a few handouts for this year that we will go through to make us all a little less bad at math. This particular handout will cover some important identities, theorems, etc. needed for events C and D for meets 1, 2, and 3.

Things to Know

Trigonometry (1C, 2C, 3C)

- Basic identities:
  - "Pythagorean identities": $\sin^2\theta + \cos^2\theta = 1$, $1 + \cot^2\theta = \csc^2\theta$, $\tan^2\theta + 1 = \sec^2\theta$
  - $\sin\theta = \cos(90° - \theta)$
  - $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$
  - $\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$, $\tan(180° - \theta) = -\tan\theta$
- Angle sum/difference identities:
  - $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
  - $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
  - $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
  - $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
  - $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
  - $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$
- Double- and Half-angle identities:
  - $\sin 2\theta = 2\sin\theta\cos\theta$
  - $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 - \cos\theta}{2}}$[1]
  - $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$
  - $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 + \cos\theta}{2}}$[2]
  - $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

[1] The $\pm$ here doesn't mean that both the positive and negative values are taken at the same time, it means that you have to determine the sign of the resulting expression separately. Also note how squaring both sides and setting $2\alpha = \theta$ here yields the reduction formulas.

[2] See 1.

  * $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$

- Trig laws: For triangle ABC, with $a = BC$, $b = AC$, and $c = AB$:

  * (Extended) Law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circumradius of ABC.
  * Law of cosines: $a^2 + b^2 - 2ab\cos C = c^2$

- Area of a triangle: Given triangle ABC, with $BC = a$, $AC = b$, and $\angle ACB = \theta$, the area of triangle ABC is $\frac{1}{2}ab\sin\theta$.

- Trig values for common angles:

Very common

$\boldsymbol{\theta}$	$\sin\boldsymbol{\theta}$	$\cos\boldsymbol{\theta}$	$\tan\theta$
$0°$	0	1	0
$30°$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45°$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	1	0	:(

(Optional) Uncommon

$\boldsymbol{\theta}$	$\sin\boldsymbol{\theta}$	$\cos\boldsymbol{\theta}$	$\tan\theta$
$15°$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$2-\sqrt{3}$
$18°$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}}$	$\sqrt{1-\frac{2}{\sqrt{5}}}$
$36°$	$\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$	$\sqrt{5-2\sqrt{5}}$
$54°$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}}$	$\sqrt{1+\frac{2}{\sqrt{5}}}$
$75°$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$2+\sqrt{3}$

- Inverse trig functions:

  * Arcsine:
    - Domain: $x \in [-1, 1]$
    - Range: $\sin^{-1}(x) \in [-\pi/2, \pi/2]$
  * Arccosine:
    - Domain: $x \in [-1, 1]$
    - Range: $\cos^{-1}(x) \in [0, \pi]$
  * Arctangent:
    - Domain: $x \in \mathbb{R}$
    - Range: $\tan^{-1}(x) \in (-\pi/2, \pi/2)$
  * Pay attention to the domains! Arcsine and Arccosine are only defined for a small set of values. Also note how $\arcsin(\sin x)$, $\arccos(\cos x)$, and $\arctan(\tan x)$ are not necessarily equal to x.

Complex Numbers (3C)

- Basic facts:
  - $i^2 = -1$
  - $i^{4k} = 1$
  - $i^{4k+1} = i$
  - $i^{4k+2} = -1$
  - $i^{4k+3} = -i$
  - $\operatorname{cis}\theta = \cos\theta + i\sin\theta$
- De Moivre's Theorem
  - For any real numbers r, x and integer n, $(r\operatorname{cis}(x))^n = r^n \operatorname{cis}(nx)$
  - Roots of unity: For a positive integer n, the solutions to the equation $x^n = 1$, also known as the nth roots of unity, are

$$1, \operatorname{cis}\frac{2\pi}{n}, \operatorname{cis}\frac{4\pi}{n}, \cdots, \operatorname{cis}\frac{2(n-1)\pi}{n}.$$

Roots of Polynomials (1D)

- Quadratic formula: The solutions to the equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$[3]

- Vieta's formulas: For a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, if its roots are $r_1, r_2, \cdots r_n$, then

$$\sum_{i=1}^{n} r_i = r_1 + r_2 + \cdots + r_n = -\frac{a_{n-1}}{a_n}$$
$$\sum_{i<j<n} r_i r_j = r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n = \frac{a_{n-2}}{a_n}$$
$$\vdots$$
$$r_1 r_2 \cdots r_n = (-1)^n \frac{a_0}{a_n}.$$

  This is used most for quadratics, where for $f(x) = ax^2 + bx + c$, $-b/a$ is the sum of the roots, and c/a is the product of the roots.

- Reciprocal polynomial: The roots of $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ are reciprocals of the roots $f^*(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$, provided that $a_0 \neq 0$.
  - *Example:* The roots of $x^2 - 5x + 6$ are 2 and 3, and the roots of $6x^2 - 5x + 1$ are $\frac{1}{2}$ and $\frac{1}{3}$.

[3] Always try to factor first, because it's easier and cleaner and saves time :P

Analytic Geometry of Lines and Circles (2D)

- Forms of line equations:
  * Point-slope: $y - y_0 = m(x - x_0)$
  * Slope-intercept: $y = mx + b$
  * Standard: $Ax + By + C = 0$
- Distance from point to line: Given line $Ax+By+C = 0$ and point (x_0, y_0), the shortest distance from the point to the line is $\dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$.
- Equation of a circle: The equation of a circle with radius r and center (h, k) is $(x - h)^2 + (y - k)^2 = 0$.
- Angle between two lines: The acute angle θ between a line with slope m_1 and a line with slope m_2 satisfies $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$.

Exponents and Logarithms (3D)

- Basics:
  * $x^{\frac{1}{n}} = \sqrt[n]{x}$
  * $x^{-n} = \frac{1}{x^n}$
  * $\log_a b = c \Leftrightarrow a^c = b$
  * $\log_a b^x = x \log_a b$
  * $\log_a bc = \log_a b + \log_a c$
  * $\log_a \frac{b}{c} = \log_a b - \log_a c$
  * $\log_a c = \dfrac{\log_b c}{\log_b a}$
  * $\log_a b = \dfrac{1}{\log_b a}$
  * $\log_{a^x} b = \dfrac{\log_a b}{x}$

Problems

Roughly sorted in order of difficulty. Problems marked with an asterisk are intended to be very challenging.

1. Let r and s be the roots of $ax^2 + bx + c$. Find $r^2 + s^2$ in terms of a, b, and c.

2. Find the eighth roots of unity.

3. Compute $\tan 7.5^\circ + \cot 7.5^\circ$.

4. Find the distance between the lines $y = 3x + 4$ and $6x - 2y = 24$.

5. In triangle XYZ, $XY = 13$, $YZ = 14$, and $\tan(\angle XYZ) = 2.4$. Find:
   (a) XZ.
   (b) The area of $\triangle XYZ$.

6. (ARML 2019 T5) Consider the system of equations

$$\begin{aligned}\log_4 x + \log_8(yz) &= 2\\ \log_4 y + \log_8(xz) &= 4\\ \log_4 z + \log_8(xy) &= 5.\end{aligned}$$

   Given that xyz can be expressed in the form 2^k, compute k.

7. (PUMaC 2015 Geometry B3) For her daughter's 12th birthday, Ingrid decides to bake a dodecagon pie in celebration. Unfortunately, the store does not sell dodecagon shaped pie pans, so Ingrid bakes a circular pie first and then trims off the sides in a way such that she gets the largest regular dodecagon possible. If the original pie was 8 inches in diameter, the area of pie that she has to trim off can be represented in square inches as $a\pi b$ where a, b are integers. What is $a + b$?

8. Triangle ABC is inscribed in circle ω. If $AB = 25$, $BC = 26$, and $AC = 27$, find the radius of circle ω.

9. Find all solutions to $(2x + 1)(2x + 8)(2x + 10)(2x + 17) = -720$.

10. (ARML 2019 T3) Compute the least positive value of t such that

$$\arcsin(\sin t),\ \arccos(\cos t),\ \arctan(\tan t)$$

   form (in some order) a three-term arithmetic progression with a nonzero common difference.

11. Find the roots to the polynomial $x^5 + x^4 + x^3 + x^2 + x + 1$.

12. Compute $\tan 15^\circ \tan 20^\circ \tan 25^\circ \cdots \tan 75^\circ$.

13. (AMC 12A 2010 #12) For what value of x does

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4(x^2) + \log_8(x^3) + \log_{16}(x^4) = 40?$$

14. * (Adapted from AHSME 1981) Let a, b, c, and d be the roots to the polynomial $x^4 - 3x + 5$. Find the monic polynomial with roots

$$\frac{a+b+c}{d^2}, \frac{a+b+d}{c^2}, \frac{a+c+d}{b^2}, \frac{b+c+d}{a^2}.$$

15. * (CMIMC 2016 Algebra #2) Suppose that some real number x satisfies

$$\log_2 x + \log_8 x + \log_{64} x = \log_x 2 + \log_x 16 + \log_x 128.$$

Find $\log_2 x + \log_x 2$.

16. * Let z_1 and z_2 be complex numbers that satisfy

$$\mathrm{Im}(z_1), \mathrm{Im}(z_2) > 0$$
$$z_1^6 = \frac{z_2^8}{64} = 64.$$

A triangle is formed in the imaginary plane using 0 and the two complex numbers in a randomly selected solution (z_1, z_2) as vertices. Find the expected value of the area of this triangle.

17. * Let a, b, and c be the roots to the equation $x^3 - 219x - 2020$. Find $a^3 + b^3 + c^3$.

18. * (AMC 12B 2018 #25) Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. What is the area of $P_1P_2P_3$?