This handout will cover some important identities, theorems, etc. needed for events C and D for meets 4 and 5 (and the state tournament, I guess).

Things to Know

Sequences and Series (4C)

- Recursive sequences: The explicit form for $a_n = \alpha a_{n-1} + \beta a_{n-2}$ is $a_n = a\lambda_1^n + b\lambda_2^n$, where λ_1 and λ_2 are the roots of the quadratic $x^2 - \alpha x - \beta$, and a and b are some constants such that the formula works for the two initial terms.
  - If $\lambda = \lambda_1 = \lambda_2$, then the explicit form is $a_n = (a + bn)\lambda^n$.
- Arithmetic sequences
  - Neighboring terms have constant difference: $a, a+b, a+2b, \cdots$
    - *Example:* $1, 4, 7, 10, \cdots$
  - Sum of all terms from a_1 to a_n is equal to $\dfrac{n(a_n + a_1)}{2}$
  - Sum of all terms from a_{k_1} to a_{k_2} with $k_2 \geq k_1$ is equal to $\dfrac{n(a_{k_1} + a_{k_2})}{2}$, where n is the number of terms, or $k_2 - k_1 + 1$.
- Geometric sequences
  - Neighboring terms have constant ratio: $a, ar, ar^2, \cdots$
    - *Example:* $2, 6, 18, 54, \cdots$
  - Sum of all terms from a_{k_1} to a_{k_2} is equal to $\dfrac{a_{k_2} - a_{k_1}}{r - 1}$, where r is the common ratio.
  - Infinite series: $\dfrac{a}{1-r} = a + ar + ar^2 + \cdots$, if $|r| < 1$
- Sum of consecutive integers: $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$
- Sum of consecutive squares: $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
- Sum of consecutive cubes: $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots n)^2 = \left(\dfrac{n(n+1)}{2}\right)^2$

Counting and Probability (4C, 5C)

- Permutations: The number of ways to choose k distinct things from n total distinct things, where order matters, is ${}_nP_k = \dfrac{n!}{(n-k)!}$
- Combinations: The number of ways to choose k distinct things from n total distinct things, where order does not matter, is ${}_nC_k = \dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$
- Important expansions:
  - $(a+b)^2 = a^2 + 2ab + b^2$
  - $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
  - Binomial Theorem: $(a+b)^n = a^n + \dbinom{n}{1} a^{n-1}b + \dbinom{n}{2} a^{n-2}b^2 + \cdots + \dbinom{n}{n-1} ab^{n-1} + b^n$
  - $\left(a + \dfrac{1}{a}\right)^2 = 2 + a^2 + \dfrac{1}{a^2}$
  - $\left(a + \dfrac{1}{a}\right)^3 = a^3 + \dfrac{1}{a^3} + 3a + \dfrac{3}{a}$
  - $\left(\displaystyle\sum_{i=1}^{n} a_i\right)^2 = \displaystyle\sum_{i=1}^{n} a_i^2 + 2\left(\displaystyle\sum_{1 \le i < j \le n} a_i a_j\right)$
- Principle of Inclusion and Exclusion (PIE)
  - Counts the number of elements that satisfies at least one of n conditions.
  - 3 condition case:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

  - General case:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n+1} \left|\bigcap_{i=1}^{n} A_i\right|$$

- Stars and Bars
  - The number of solutions to $a_1 + a_2 + \cdots a_k = n$, where a_i, k, and n are all positive integers, is $\binom{n-1}{k-1}$.
  - The number of solutions to $a_1 + a_2 + \cdots a_k = n$, where k and n are positive integers and the a_i are non-negative integers, is $\binom{n+k-1}{k-1}$.

Conic Sections (4D)

- Circles: See handout 1.
- Ellipses
  - ∗ Set of points with a constant sum of distances from two fixed points (the foci).
  - ∗ Has two axes:
    - – Major axis: longest diameter of the ellipse
    - – Minor axis: shortest diameter of the ellipse
  - ∗ Area is $ab\pi$, where a and b are half of the lengths of the major and minor axes.
  - ∗ If $a \geq b > 0$, then $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ is the equation for an ellipse with center (h, k), and foci at $(h \pm c, k)$, with $c^2 = a^2 - b^2$, and has a major axis of length $2a$ and a minor axis of length $2b$.
  - ∗ If $a \geq b > 0$, then $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ is the equation for an ellipse with center (h, k), and foci at $(h, k \pm c)$, with $c^2 = a^2 - b^2$ and has a major axis of length $2a$ and a minor axis of length $2b$.
- Parabolas
  - ∗ Set of points that are equidistant from a point (the focus) and a line (the directrix)
  - ∗ The equation for a parabola with focus $(h, k + p)$ and directrix $y = k - p$ is $(x - h)^2 = 4p(y - k)$.
  - ∗ The equation for a parabola with focus $(h + p, k)$ and directrix $x = h - p$ is $(y - k)^2 = 4p(x - h)$.
- Hyperbolas
  - ∗ Set of points with a constant difference of distances from two fixed foci.
  - ∗ $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ is the equation for an ellipse with center (h, k), asymptotes at $y - k = \pm\frac{b}{a}(x - h)$ and foci at $(h \pm c, k)$, with $c^2 = a^2 + b^2$
  - ∗ $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ is the equation for an ellipse with center (h, k), asymptotes at $y - k = \pm\frac{a}{b}(x - h)$ and foci at $(h, k \pm c)$, with $c^2 = a^2 + b^2$

Problems

Roughly sorted in order of difficulty. Problems marked with an asterisk are intended to be very challenging.

1. Find the explicit form of the Fibonacci sequence $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. *Note*: This is also known as Binet's formula.

2. Jennifer has 9 identical tennis balls she wants to put into 4 distinct, colored bins. Provided that every bin has at least one ball in it, compute the number of ways she can organize her tennis balls.

3. (2016 AMC 12A #11) Each of the 100 students in a certain summer camp can either sing, dance, or act. Some students have more than one talent, but no student has all three talents. There are 42 students who cannot sing, 65 students who cannot dance, and 29 students who cannot act. How many students have two of these talents?

4. (2002 AIME I #1) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. An ellipse in the plane is tangent to the x-axis at $(8, 0)$ and tangent to the line $y = x$. Given that the ellipse is inside Quadrant I (except the point of tangency to the x-axis) and that the minor axis has length 1, find the point shared by the ellipse and the line $y = x$.

6. Let the parabola $(y - 2)^2 = x$ be tangent at two points to a circle centered at the origin. Find the area of this circle.

7. (HMMT February 2013 Algebra #2) Let $\{a_n\}_{n\geq 1}$ be an arithmetic sequence and $\{g_n\}_{n\geq 1}$ be a geometric sequence such that the first four terms of $\{a_n + g_n\}$ are 0, 0, 1, and 0, in that order. What is the 10th term of $\{a_n + g_n\}$?

8. $*$ Compute $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots$

9. $*$ (2001 AIME I #5)An equilateral triangle is inscribed in the ellipse whose equation is $x^2 + 4y^2 = 4$. One vertex of the triangle is $(0, 1)$, one altitude is contained in the y-axis. Find the length of each side of the triangle.

10. $*$ Compute $\displaystyle\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{a^3 + 3ab^2}{(a + b)^3 2^{a+b}}$.