Problem Sheet #1

I've got 99 problems
and now Jay-Z is one

Barack Obama
White House Correspondents' Dinner
April 27, 2013

Problems are roughly sorted in order of difficulty. Problems marked with asterisks are meant to be challenging.

1. What are the solutions to the equation $x^2 - 12x + 35 = 0$?
2. Find the area of the solution set to the equation $x^2 + x + y^2 - y + \frac{1}{4} = 0$.
3. How many two-digit positive integers are divisible by 4 or 7, but not both?
4. What is the solution to the equation $\ln x = \log_{\ln x} x$?
5. Equilateral triangle ABC is inside square $ADEF$, such that B lies on side DE and C lies on side EF, as shown in Figure 1. Find $[ABC]/[ADEF]$.
6. Find the last digit of 2^{2020}.
7. How many distinct triangles can be made using sides with distinct side lengths from the set $\{1, 2, 3, \ldots, 12\}$?
8. Find the area of $\triangle HXY$, where $ABCD$ is a square, $AH = GC = CF = EA = 1$, and $HD = DG = FB = BE = 2$, as shown in Figure 2.

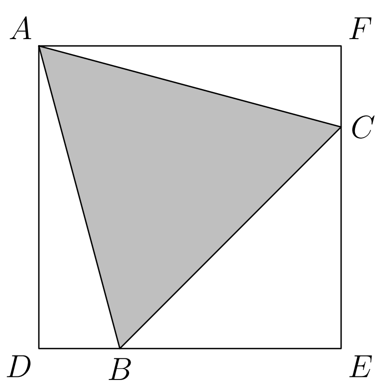

Figure 1

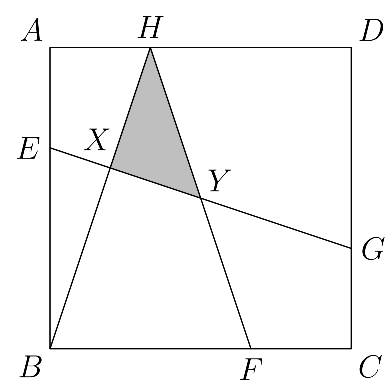

Figure 2

9. (2018 AMC 8) How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?
10. Let A, B, C, D, and E be digits such that the four-digit number $ABC6$ is equal to 11 times the two-digit number DE. Find the five-digit number $ABCDE$.

11. 18! is equal to $6,402,373,705,7\underline{ab},000$. Find the product $a \cdot b$.

12. Find the remainder when 2020^{19} is divided by 7.

13. Each face of a cube is painted either red or blue. Find the number of ways to paint the cube, if two paintings that can be obtained through a rotation are considered identical.

14. Define the base-2 *iterated logarithm* of x to be

$$\log_2^* x = \begin{cases} 0 & \text{if } x \leq 1, \\ 1 + \log_2^*(\log_2 x) & \text{if } x > 1. \end{cases}$$

Find the smallest integer n such that $\log_2^* n = 5$.

15. Find all roots to $x^4 - 2x^3 - 7x^2 + 8x + 12$.

16. $*$ (2014 AIME II) Let $f(x) = (x^2 + 3x + 2)^{\cos(\pi x)}$. Find the sum of all positive integers n for which

$$\left| \sum_{k=1}^{n} \log_{10} f(k) \right| = 1.$$

17. $*$ $\triangle ABC$ has side lengths $AC = 3$, $AB = 4$, and $BC = 5$, and has incenter D. Circles Γ_1, Γ_2, and Γ_3 are drawn inside triangle ABC such that all three circles pass through D, Γ_1 is tangent to AB and AC, Γ_2 is tangent to AB and BC, and Γ_3 is tangent to AC and BC, as shown in Figure 3. Find the sum of the areas of circles Γ_1, Γ_2 and Γ_3.

18. $*$ The expression $\sqrt[3]{25 + \sqrt{a}} + \sqrt[3]{25 - \sqrt{a}}$ is exactly equal to 5. What is the value of a?

19. $*$ Find the remainder when 2020^{2019} is divided by 77.

20. $*$ Compute $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^n}$.

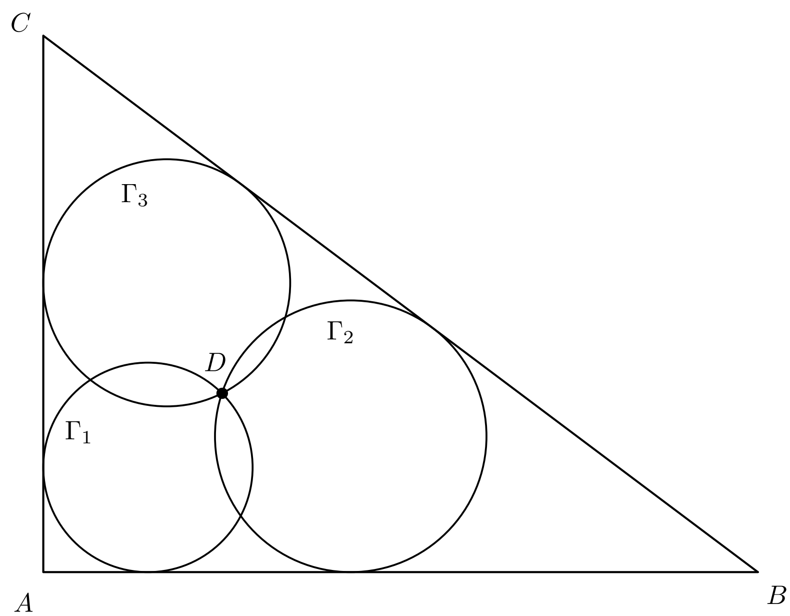

Figure 3